Faithfully Flat Descent

Def: Let $\mathcal{C}$ be a category with fiber products. A morphism $f: Y \to X$ is a strict epimorphism if for all $Y \times_X Y \rightrightarrows Y \xrightarrow{f} X$, $\mathrm{Hom}(X, Z) \xrightarrow{f^*} \mathrm{Hom}(Y, Z) \rightrightarrows \mathrm{Hom}(Y \times_Z Y, Z)$ has f^* as the equalizer, $\forall Z$.

Example: A surjection of schemes need not be a strict epimorphism. Consider:

$$Y = \mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, k[\varepsilon]/\varepsilon^2 = X, \quad \text{and set } Z = X.$$

Then $\mathrm{Hom}(X,X) \to \mathrm{Hom}(Y,X)$ is not injective, and so cannot be an equalizer.

Thm (Descent for morphisms): A faith. flat $f: Y \to X$ of finite type is a strict epimorphism.

Prop: If f is a faith. flat morphism of rings, then the sequence of A-modules:

$$(*) \quad 0 \to A \xrightarrow{f} B \xrightarrow{d^0} B \otimes_A B \xrightarrow{d^1} B \otimes_A B \otimes_A B \xrightarrow{d^2} \cdots \quad \Big\} \text{ Amitsur complex.}$$

where $e_i(b_0 \otimes \cdots \otimes b_{r-1}) = b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}$, and $d^{r-1} = \sum_{i=0}^{r} (-1)^i e_i$, is exact.

Proof: Prop 2.18 in EC. $d^2 = 0$ is standard. To see exactness, first assume $f: A \to B$ has a left inverse $g: B \to A$. Define $R_r: B^{\otimes r+2} \to B^{\otimes r+1}$ by

$$R_r(b_0 \otimes \cdots \otimes b_{r+1}) = g(b_0)\, b_1 \otimes \cdots \otimes b_{r+1}.$$

One can check $R_{r+1} d^{r+1} + d^r R_r = \mathrm{id}$. So the complex is contractible, and so it is exact.

Now it suffices to show $B \otimes_A (*)$ is an exact complex by faithful flatness. But this complex is:

$$0 \to B \to \underbrace{B \otimes_A B}_{= B'} \to \cdots \qquad b \longmapsto b \otimes 1.$$

This has a left inverse by $b \otimes 1 \mapsto b$.
$\Rightarrow$ We are done. ∎

We claim that if M is an A-module, then $M \otimes_A (*)$ is still exact. Indeed since f is faithfully flat, it suffices to check that after tensoring with B, $B \otimes_A M \otimes_A (*)$ is still exact. But $(*) \otimes_A B$ is the Amitsur complex for the map $B \to B \otimes_A B$, and noting that this is contractible gives the claim by homological algebra.

Now we prove the theorem (Thm. 2.17 in EC).

Proof: Note this is the same as saying the Grothendieck topology of faithfully flat maps of finite type is subcanonical. Consider

$$\begin{array}{ccc} Y \times Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{f} & X \end{array}$$

WTS: Given any scheme Z, the diagram $\mathrm{Hom}(X, Z) \xrightarrow{f^*} \mathrm{Hom}(Y, Z) \rightrightarrows \mathrm{Hom}(Y \times Y, Z)$ is an equalizer diagram.

We first look at the affine case. Set $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$. Then the diagram is: $\operatorname{Hom}(C,A) \longrightarrow \operatorname{Hom}(C,B) \rightrightarrows \operatorname{Hom}(C, B\otimes B)$, which gives

$$\left.\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & B\otimes_A B \\ & & C \arrow[u] \arrow[ul, dashed, "\exists !"] & \end{tikzcd}\right\}$$

By the Amitsur complex. $\Rightarrow$ Done.

Now suppose only Z is arbitrary. Then we want:

$$\begin{tikzcd} Y\times_X Y \arrow[d, shift left] \arrow[d, shift right] & \\ Y \arrow[d, "f"'] \arrow[dr, "h"] & \\ X \arrow[r, dashed, "\exists ! g"'] & Z \end{tikzcd}$$

first show unique. Suppose we have $g_1 + g_2$ which work ($g_1\circ f = g_2 \circ f$). Since f is surjective, $g_1 = g_2$ set-theoretically. It's enough now to show they are locally the same. Let U contain $g_1(x) = g_2(x)$ for some $x\in X$, then $g_1^{-1}(U) = g_2^{-1}(U) \subset X = \operatorname{Spec} A$. We can assume $U = X_a = \operatorname{Spec} A_a$, hence $f^{-1}(X_a) \subset Y$ is $Y_b = \operatorname{Spec} B_b$. By case (a), $g_1|_{X_a} = g_2|_{X_a} \Rightarrow g_1 = g_2$ by glueing local ones.

Let $x\in X$, and $y \in f^{-1}(x)$. Choose an affine open $U \subseteq Z$ containing $h(y)\in Z$. Now we claim $f(h^{-1}(U)) \subset X$. Since f is flat and finite type, this set is open. Hence we can choose an affine open V of x inside $f(h^{-1}(U))$. Since $X = \operatorname{Spec} A$, choose $V = \operatorname{Spec} A_a = X_a$. Then $f^{-1}(X_a) = Y_b$, and we only need to check $h(Y_b)$. But this follows from fiber products, and we are in case (a).

The general case is in the text. ∎

<u>Thm (FGA explained):</u> "fptf (f. flat fin. type) $\subset$ fpqc": A covering $f: Y \to X$ is a faithfully flat map such that $\forall$ quasi-compact $U \subset X$, $U = f($q.c. open in $X)$. Moreover, fpqc is subcanonical.

<u>Exercise 6:</u> Show that $\operatorname{Spec} k[t] \to \operatorname{Spec} k[t^3, t^5]$ is an epimorphism, but not a strict epimorphism.

Descending Modules

Let $f: A \to B$ be f. flat, M an A-module. Setting $M' = B\otimes_A M = f_* M$, by the Amitsur complex:

$$A \xrightarrow{f} B \underset{e_1}{\overset{e_0}{\rightrightarrows}} B\otimes_A B \;\substack{\longrightarrow\\ \longrightarrow \\ \longrightarrow}\; B\otimes_A B \otimes_A B \;----,$$

$e_{0*} M' = (B\otimes_A \cancel{B})\otimes_{\cancel{B}} M' = B\otimes_A M'$ w/ $(b_0\otimes b_1)(b\otimes M') = b_0 b \otimes b_1 M'$.
$e_{1*} M' = (\cancel{B}\otimes_A B)\otimes_{\cancel{B}} M' = M'\otimes_A B$ w/ $(b_1\otimes b_2)(M'\otimes b) = b_1 M' \otimes b_2 b$.

So we have an isomorphism $\phi: e_{1*}M' \to e_{0*}M'$ via $(b\otimes m)\otimes b' \longmapsto b\otimes(b'\otimes m)$.
We can recover M from ϕ by $M = \{ m\in M' \mid 1\otimes m = \phi(m\otimes 1)\}$.

Now given some M', under what conditions can we recover M? Such conditions are called descent data. For $M' = B\otimes_A M$ for some M, we surely need an isomorphism $\phi: e_{1*}M' \xrightarrow{\sim} e_{0*}M'$, such that if

$$\phi_1 = B\otimes\phi : B\otimes_A M'\otimes_A B \longrightarrow B\otimes_A B\otimes_A M'$$
$$\phi_2 = \text{"}\phi\otimes B\otimes\phi\text{"} : M'\otimes B\otimes B \longrightarrow B\otimes B\otimes M'$$
$$\phi_3 = \phi\otimes B : M'\otimes B\otimes B \longrightarrow B\otimes M'\otimes B$$

Then $\phi_2 = \phi_1 \circ \phi_3$. This is called the cocycle condition. It turns out this is enough to get an M. How to find M? Put

$$M = \{ m \in M' \mid 1 \otimes m = \phi(m \otimes 1) \},$$

and show $B \otimes_A M \cong M'$. This is rather technical, but the upshot is:

Thm: Given a f.flat $f: A \to B$ of rings, we get an equivalence of categories:
$A\text{-Mod} \longleftrightarrow \{ \text{Descent data } (\phi, M') \text{ w/ } \phi_2 = \phi_1 \phi_3 \}$.

Prop: Let $f: Y \to X$ be f.flat and quasi-compact map of schemes. Then giving a quasi coherent sheaf M on X is the same as giving a quasi coherent sheaf M' on Y with $\phi: p_1^* M' \to p_2^* M'$, $p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi)$ in the diagram:

$$Y \times_X Y \times_X Y \overset{p_{31}}{\underset{p_{21}}{\overset{\longrightarrow}{\underset{p_{32}}{\longrightarrow}}}} \!\!\!\! \longrightarrow Y \times_X Y \overset{p_1}{\underset{p_2}{\rightrightarrows}} Y \xrightarrow{f} X.$$

Project 1: Let $f: Y \to X$ be f. flat. Assume Y is integral (resp. normal, regular). Then so is X.

Project 2: Let $g: X' \to X$ be f.flat and quasi-compact. Then in

$$\begin{tikzcd} X' \times Y \arrow[r] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

if f' is q. compact (separated, fin. type, proper, open immersion, affine, finite, q. finite, flat, smooth, étale), then so is f. (Do any 3).

Project 3 (3 exercises): Not worth it.

Consider G a group acting on a top. space Y. Then: $G \times Y \overset{\pi}{\underset{\sigma}{\rightrightarrows}} Y$, and this can be extended to:

$$\begin{tikzcd} G \times Y \times Y \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \times Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[l, bend right, "s"'] \end{tikzcd} \qquad s(y) = (e, y)$$

So a sheaf F on Y is G-equivariant if given $\Theta: \pi^* F \to \sigma^* F$ and $s^* \Theta: s^* \pi^* F \xrightarrow{id} s^* \sigma^* F$, we can take $X = Y/G$ w/ $f: Y \to X$, then a sheaf on X with the above "descent data". If $f: Y \to X$ is a principal homogeneous G-space ($X = \cup U_i$, $f^{-1}(U_i) \cong G \times U_i$ is an iso. of G-spaces), then we have an equivalence of categories $Sh(X) \cong Sh_G(Y)$.